A CLASS OF INVERSE PROBLEMS AND METHODS OF THEIR SOLUTION

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The formulation is given of a class of inverse problems which includes a number of inverse thermophysical problems as a particular case. Algorithms of the solution of such problems are considered and their qualitative characteristic is given.

1. Direct problems (including thermophysical problems) often admit of the following formulation: Given two operators A_1 and A_2 acting from the same Banach space U into the Banach spaces F_1 and F_2, respectively, find the element $\bar{u} \in U$ satisfying the system of equations

$$A_1 \bar{u} = f_1, \qquad A_2 \bar{u} = f_2, \tag{1}$$

where f_1 and f_2 are given elements from F_1 and F_2.

The problem (1) is called correctly formulated (correct) if:

1) the solution $\bar{u}$ of the problem (1) exists for any $f_1 \in F_1$ and $f_2 \in F_2$;

2) it follows that u = v from the equalities $A_1 u = A_1 v$, $A_2 u = A_2 v$, u, v $\in$ U;

3) infinitesimal variations of the elements f_1 and f_2 result in infinitesimal variations of the solution $\bar{u}$ (in the appropriate spaces).

Conditions 1)-3) are known to be satisfied if $A_1 U = F_1$, $A_2 U = F_2$ and the operators A_1 and A_2 are linear and the following a priori estimate holds:

$$\| u \| \leqslant k (\| A_1 u \|_1 + \| A_2 u \|_2), \quad \forall u \in U, \quad k > 0.$$

The solution $\bar{u}$ of the problem (1) is evidently a function of the elements f_1 and f_2, i.e., $\bar{u} = \bar{u}(f_1, f_2)$. The first equation of (1) can be interpreted as the fundamental equation and the second, as the boundary and (or) initial conditions.

The element $f_2 = x$ is considered unknown in inverse problems. We have for a fixed element f_1: $\bar{u} = \bar{u}(f_1, x) \equiv \bar{u}_1(x)$, $\forall x \in F_2$, i.e., we have a mapping of the space F_2 into U. Its image does not certainly coincide with U, and therefore, assigning some u $\in$ U does not always result in the determination of x from the equation $\bar{u}_1(x) = u$.

Complete information about the element $\bar{u}_1(x)$ is usually redundant to the determination of x. In this connection, the operator B acting from U into the Banach space Y is defined in U, and the element $\bar{y} = B\bar{u}_1(x)$ is measured. The operator B is often the trace operator of the function u $\in$ U on manifolds of lower dimension (or a system of functionals of u). The final formulation of the inverse problem is to solve the operation equation

$$Kx = y, \tag{2}$$

where the operator $K = B\bar{u}_1(\cdot)$ acts from F_2 into Y.

We later consider that $f_1 = 0$ and the operators A_1 and A_2 are linear. In this case the operator K will also be linear.

Example. Let the direct problem consist of determining the temperature $u(z, t)$ of a semi-infinite homogeneous rod. As is known, its satisfies the following equations:

$$A_1 u \equiv \left(\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial z^2} , \ u \big|_{t=0} \right) = f_1 = (\bar{f}(z, t), \ \varphi(z)),$$

$$0 \leqslant z < +\infty, \quad 0 \leqslant t \leqslant T,$$

$$A_2 u \equiv u \big|_{z=0} = x(t), \quad 0 \leqslant t \leqslant T.$$

The inverse problem consists of determining the temperature $x(t)$ on the boundary of the rod $z = 0$. The operator B can be defined by different methods. For example, $Bu(z, t) = u(\bar{z}, t)$, $0 < \bar{z} < +\infty$ or $Bu(z, t) = u(z, T)$. An L_2 type space is often selected as the space Y.

The problem of a historical climate is comprised in an analogous scheme.

As a rule, inverse problems are incorrect.

2. Let us call the space $F_2 = X$ the space of states, and the space Y the space of appearances (or responses).

Equation (2) is the mathematical model of the physical process being studied and expresses the cause–result relationship between the desired state x and its appearance y.

The following problems occur:

a) estimate the adequacy of the model (2) by means of the measured appearance $\tilde{y}$ (it is considered that the measurements are regular; i.e., correspond to the physical process being studied);

b) indicate an approximate method of determining states close to the theoretically possible $\bar{x}$ for an adequate model.

The importance of their solution is indubitable.

Let us first examine problem a).

Following [4], let us call the model (2) consistent if the set of admissible states $\tilde{X} = \{x : \| Kx - \tilde{y} \| \leqslant \delta \}$, where $\delta : \| \bar{y} - \tilde{y} \| < \delta$, is not empty for all δ. Let us set $\mu = \inf_{x \in X} \| Kx - \tilde{y} \|$. We call the quantity μ the measure of incompatibility of (2). If $\mu \ll \| \bar{y} \|$, then the model (2) is called adequate $(\bar{y}$-adequate). If $\mu = 0$, we then call the model (2) compatible. If an element $\hat{x} \in X$ exists for which $\| K\hat{x} - \bar{y} \| = \mu$, then the model (2) is called solvable. Let us note that the model can be compatible but not solvable. A solvable model (2) is called single-valued if the element $\hat{x}$ is unique.

It is easy to show that for the consistency of the model (2) it is necessary and sufficient that $\mu = 0$, i.e., the model (2) be compatible. If $\mu = 0$, then the model (2) is ε-solvable in the sense that the sets $X_\varepsilon = \{x \in X : \| Kx - \bar{y} \| \leqslant \varepsilon \}$ are not empty for any $\varepsilon > 0$, i.e., in this case the model (2) is solvable in practice.

If $\mu = 0$, $\forall \ \bar{y} \in Y$, then by virtue of the above the model (2) is consistent for any $\bar{y} \in Y$. We call such a model absolutely consistent. It is easy to prove that it is necessary and sufficient for this that the closure of the image Q_K of the operator K coincide with Y, i.e., $Q_K = Y$. The model (2) is then solvable in practice for all $\bar{y} \in Y$ (see [15]).

In connection with the above, it seems reasonable to consider absolutely consistent mathematical models.

Let the model (2) be solvable. Will it be single-valued? Not in the general case. The fallacy is extended that uniqueness of the solution of the inverse problem follows from the single-valuedness of the direct problem. There is no necessity to prove the inconsistency of this statement. The problem of uniqueness of inverse problems is an important aspect of the general theory of the solution of incorrect problems. It is analyzed most completely by M. M. Lavrent'ev and his pupils.

Let us set $\tilde{\mu} = \inf_{x \in X} \| Kx - \tilde{y} \|$. It is easy to show [15] that $| \mu - \tilde{\mu} | \leqslant \| \bar{y} - \tilde{y} \| < \delta$. It hence follows that $\mu \approx \tilde{\mu}$. If $\tilde{\mu} \ll \| \bar{y} \|$ then evidently also $\mu \ll \| \bar{y} \|$ (for small δ).

A stable evaluation of $\tilde{\mu}$ in conformity with the definition is impossible for reasons whose explanation would require considerable space. Let us limit ourselves to a referral to [5], where this question is resolved with a high degree of rigor and an effective algorithm is proposed for the evaluation of approximations to μ which would assure an estimate of the adequacy of the model.

Let us note that the solution of the problem a) is especially important for the analysis of new mathematical models.

3. Let us turn to the solution of problem b). Let us assume the model (2) to be absolutely consistent. Since the problem (2) is incorrect, the diameter of the set $\tilde{X}$ does not tend to zero as $\delta \to 0$. Meanwhile, this set includes all reasonable states x consistent with the measured $\tilde{y}$. It is necessary to formulate the selection rule for such states. The concept of a "reasonable" state has not been defined. In the general case they may be realizable (or physically accomplishable) states. Needed for their selection is contraction of the domain of admissible states because of the involvement of additional a priori constraints.

This can be accomplished structurally as follows. A functional $\Omega(x) \geq 0$, $x \in D_\Omega \subseteq X$ is defined on the part X such that $\Omega(0) = 0$ and

$$\Omega\left(\frac{x_1 + x_2}{2}\right) < \frac{1}{2}\Omega(x_1) + \frac{1}{2}\Omega(x_2), \quad \forall x_1 \neq x_2.$$

A certain sample state $x^* \in D_\Omega$ is chosen. Then the functional $\Omega_0(x) \equiv \Omega(x - x^*)$ required can be interpreted as a penalty functional and the value $\Omega_0(x)$ as the penalty for the deviation of x from the given state x^*. Naturally $\overline{\underset{R>0}{\cup X_R}} = X$, where the set is $X_R = \{x \in D_\Omega : \Omega_0(x) \leq R\}$. Here the quantity R characterizes the level of the penalty. The functional $\Omega_0(x)$ can also characterize the complexity (smoothness, mathematically) of the state $z = x - x^*$. Then X_R is a set of states not exceeding the admissible level R in complexity. Often $\Omega(x) = \|x\|$.

The following selection rule becomes clear from the above: it is necessary to select admissible states for which the complexity does not exceed the complexity of the proposed true state. This can be achieved by different means. We now present a few of them.

Method I [6]. Both δ and R are known. If the set $X_{R,\delta} = X_R \cap \tilde{X}$ is not empty, then any element from $X_{R,\delta}$ can be taken as the supporting solution of the problem (2). Mathematically the problem reduces to determining the common points of two sets, which have been studied well in the theory of convex programming.

Method II. Both δ and R are known. The compromise state $\hat{x}_R$ is selected from the condition

$$\hat{x}_R \subset D_\Omega : \|Kx - \tilde{y}\|^2 + \frac{\delta^2}{R^2}\Omega_0^2(x) - \min. \tag{3}$$

Since

$$\|K\hat{x}_R - \tilde{y}\|^2 + \frac{\delta^2}{R^2}\Omega_0^2(\hat{x}_R) \leq \|K\bar{x} - \tilde{y}\|^2 + \frac{\delta^2}{R^2}\Omega_0^2(\bar{x}) \leq \delta^2 + \frac{\delta^2}{R^2}R^2 = 2\delta^2,$$

i.e., the found state is $\hat{x}_R \in X_{\sqrt{2}R, \sqrt{2}\delta}$. This shows that the Method II is close to the Method I but its realization is simpler.

A common disadvantage of Methods I and II is the necessity to give both δ and R simultaneously.

Method III (The Residual Method [7, 8]). This consists of selecting that state x_δ among the admissible states which possess minimal complexity, i.e.,

$$x_\delta \in D_\Omega : \Omega_0(x) - \min. \tag{4}$$

Evidently $\|Kx_\delta - \tilde{y}\| \leq \delta$, $\Omega_0(x_\delta) \leq \Omega_0(\bar{x}) \leq R$, i.e., $x_\delta \in X_{R,\delta}$. Knowledge of R is not necessary to the realization of (4).

Method IV (The Quasisolution Method [9]). Let us give the complexity level R and let us define the state $\tilde{x}_R$ from the condition

$$\tilde{x}_R \in X_R : \|Kx - \tilde{y}\| - \min. \tag{5}$$

It is easy to see that $\Omega_0(\tilde{x}_R) \leq R$, $\|K\tilde{x}_R - \tilde{y}\| \leq \delta$, i.e., $\tilde{x}_R \in X_{R,\delta}$.

Thus, Methods III and IV can be considered the realization of the Method I.

Under definite conditions, the solution of problems (6) and (7) by the method of Lagrange multipliers reduces to a problem of absolute minimization. The selection of the Lagrange multiplier is accomplished effectively on the basis of algorithms described in [10]. A number of other numerical methods of solving incorrect problems has been examined in [15].

4. The so-called method of trials (adjustments) is widespread in the practice of solving inverse problems. The possible states x_i, $i = 1, 2, \ldots, s$ are usually given. Solving the direct problem by the scheme given in Sec. 1, the elements $y_i = Kx_i$, $i = 1, 2, \ldots, s$ are determined. Preference is given to that state x_i for which $\|y_i - \tilde{y}\| - \min$.

The simplicity of its realization wins out in this approach. But could it be considered effective? No, if the possible states are not selected from a previously assigned bounded class of admissible states. Here the intuition of the interpreter is usually assumed, i.e., an objective solution of the inverse problem becomes impossible.

An objective trial method on the basis of the method of quasisolutions has been proposed in [11].

5. Let us examine the selection of the functional $\Omega(x)$. If the operator K is given inaccurately (and this is necessary for the realization of any method on an electronic computer), then it is easy to give an estimate of the form

$$\|\tilde{K}x - Kx\| \leqslant \nu(\tilde{K}, K)\Omega(x - x^*),$$

where $\nu(\tilde{K}, K) \to 0$ as $\tilde{K} \to K$ and is independent of x, but $\Omega(x)$ is often the norm of some derivative of the function x. It is then natural to set $\Omega_0(x) \equiv \Omega(x - x^*)$. This question is elucidated more completely in [12].

The functional is usually $\Omega(x) = (Cx, x)^{1/2}$ for a finite-numerical realization, where $x = (x_1, x_2, \ldots, x_n)$ is the desired vector solution and C is a positive-definite matrix.

Then the Method II reduces to solving the system of equations

$$\frac{\delta^2}{R^2}C(x - x^*) + K^T Kx = K^T \tilde{y}. \tag{6}$$

The element x^* can be interpreted as the "mean" among the possible states which is obtained from direct measurements, for example, and the matrix $R^2 C^{-1}$ as the "correlation" matrix expressing the degree of dependence of components of the vector x (smoothness). It is easy to see that the method II is a deterministic analog of the Bayes regularization method proposed in [13]. In this connection let us turn attention to the error in the viewpoint of the authors of [14], who contrasted randomized schemes of the regularization method to the deterministic schemes.

In conclusion, let us note that the Methods I-IV possess optimality properties in the sense of unimprovability of the order of the accuracy of the approximations they provide [15].

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